Exercise 12

Convert each of the following Volterra integral equation in 9–16 to an equivalent IVP:

$$u(x) = \sin x - \int_0^x (x-t)u(t)\,dt$$

Solution

Differentiate both sides with respect to x.

$$u'(x) = \cos x - \frac{d}{dx}\int_0^x (x-t)u(t)\,dt$$

Use the Leibnitz rule to differentiate the integral.

$$u'(x) = \cos x - \left[\int_0^x \frac{\partial}{\partial x}(x-t)u(t)\,dt + (0)u(x)\cdot 1 - (x)u(0)\cdot 0\right]$$

$$u' = \cos x - \int_0^x u(t)\,dt$$

Differentiate both sides with respect to x again.

$$u'' = -\sin x - \frac{d}{dx}\int_0^x u(t)\,dt$$

$$u'' = -\sin x - u(x)$$

$$u'' + u = -\sin x$$

The initial conditions to this ODE are found by plugging in $x = 0$ into the original integral equation,

$$u(0) = \sin 0 - \int_0^0 (x-t)u(t)\,dt = 0,$$

and the formula for u',

$$u'(0) = \cos 0 - \int_0^0 u(t)\,dt = 1.$$

Therefore, the equivalent IVP is

$$u'' + u = -\sin x,\ u(0) = 0,\ u'(0) = 1.$$